Discrete Dynamical Systems

The Tent Map and the Cantor Set

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Mandelbrot Set

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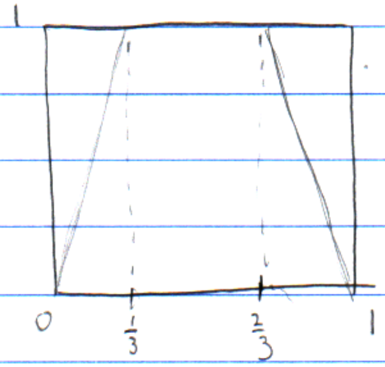
$\frac{1}{3}$ tent map and the Cantor set – Discrete Dynamical Systems

'Review' – We are examining the set

$$\Lambda_{1/3} = \{ x \in [0,1],\ x_n = T^n_{1/3}(x) \text{ is in } [0,1] \text{ for all } x \}$$

which are all the points on the x-axis of the function



$$x_{n+1} = T_{\frac{1}{3}}(x_n) = \begin{cases} 3x_n & \text{if } 0 \le x_n \le \frac{1}{3} \\ -3x_n + 3 & \frac{2}{3} \le x_n \le 1 \\ \infty & \frac{1}{3} < x < \frac{2}{3} \end{cases}$$

that never enter the interval $(\frac{1}{3}, \frac{2}{3}) \rightarrow$ "bad region"

- If we erase all the intervals that go to the bad region after one iteration, 2 iterations etc.
  - 1st erase bad region → middle third
  - as we've shown, we erase the middle third of every remaining set
  - so we're left with the Cantor Set

Ternary expansion:

- base 3, uses 0's 1's 2's
- In our tent map: 1 corresponds to the bad region
- So numbers in our set have ternary expansions consisting of only 0's and 2's

Cardinality

- different ideas of infinity
- for example, there are more numbers in [0,1] than there are natural numbers (probably won't prove in our presentation)
- uncountable: one-to-one correspondence with numbers in [0,1]
- countable = natural numbers

Claim: $\overbrace{\text{Cantor set}}^{\Delta_{1/3}}$ has the same cardinality as the interval $[0,1]$
[one-to-one correspondence]

Proof: All numbers with ternary expansions consisting only of 0's and 2's are in the set $\Delta_{1/3}$

- Expand every number $[0,1]$ into binary (so they consist of only 0's and 1's)
- Now change every '2' in the ternary expansion of $\Delta_{1/3}$ into a '1'
- So now every number in $\Delta_{1/3}$ corresponds to a binary expansion of a number in $[0,1]$ and we are done

\- Length of space removed in Cantor set:

$$\frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \dots = \sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{\frac{1}{3}}{1-\frac{2}{3}} = 1$$

so we are removing the whole line of length 1

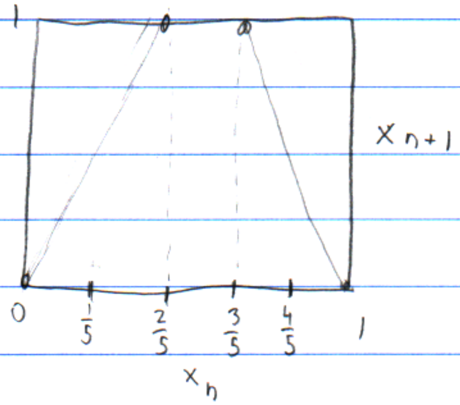
$\frac{1}{3}$ Cantor set has <u>measure</u> 0

\- <u>measure</u> - technical mathematical definition ... basically assigns a "size" to a set. Refines inexact ideas like 'length' (measures are used in integration in Calculus)

A general Cantor set has a positive measure or a measure of 0

$\frac{1}{5}$ Cantor Set / Tent Map

$$T_{\frac{1}{5}} \begin{cases} 5x & 0 \le x \le \frac{2}{5} \\ 5x-5 & \frac{3}{5} \le 1 \le 1 \\ \infty & \frac{2}{5} < x < \frac{3}{5} \end{cases}$$



- remove the middle 5th
- However, if we add up the space removed:

$$\frac{1}{5} + \frac{2}{25} + \frac{4}{125} + \ldots = \sum_{n=0}^{\infty} \frac{1}{5}\left(\frac{2}{5}\right)^0 = \frac{\frac{1}{5}}{\frac{3}{5}} = \frac{1}{3}$$

So there is still $\frac{2}{3}$ left

- Cantor sets can have either 0 or positive measure
- However, they still contain no intervals
- Cardinality is same as interval $[0,1] \rightarrow$ can be proven in similar way as with $T_{1/3}$

Differences between $\frac{1}{3}$ and $\frac{1}{5}$ Cantor Map

- '1' = total space removed from $\frac{1}{3}$ set, '$\frac{1}{3}$' = total space removed from $\frac{1}{5}$ set
- In $\frac{1}{3}$ set, we always remove the <u>middle</u> third of remaining interval
  In $\frac{1}{5}$ set, we first remove middle fifth, but then we aren't removing the middle of intervals
- $\frac{1}{5}$ set is not symmetrical like $\frac{1}{3}$ and has positive measure
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Similarity of $\frac{1}{5}$ and $\frac{1}{3}$ Cantor sets

- both have uncountable points (one-to-one correspondence w/numbers in [0,1])
  - can be proven as we did with $\frac{1}{3}$, but using base 5 and base 4 expansions
- No intervals
- Dense (no matter how small the interval, there is a point from Cantor Set in it)

Mandelbrot Set

- Our program generates the Mandelbrot Set and graphs it.
- Definition: all complex numbers $c = a + bi$ such that

$$X_0 = 0$$

$$X_{n+1} = X_n^2 + c$$

never diverges

- the graph is a fractal
- what this means: picture repeats itself, you can zoom into a part of the graph to see a miniature version

$\frac{1}{3}$ Cantor set and tent map

G+V	Review 1 min
Andrea	Cardinality
Isabel	Ternary expansion Proof that Cantor set has same cardinality as [0, 1]
Valeria	Space removed from $\frac{1}{3}$ Cantor set, geometric sequence
Andrea	Introduce idea of measure $\frac{1}{5}$ Cantor set and tent map Gaby
Valeria	differences from $\frac{1}{3}$ (space removed, symmetry) similarity (no intervals, uncountable)
all	Mandelbrot (fractal, explain generation)

$z_{n+1} = z_n^2 + c$ [c is a complex # $a+bi$]